CO6 “Introduction to Computational Neuroscience”

Lecturer:
Boris Gutkin
Ecole normale supérieure
29 Rue d’Ulm, 2nd Floor)
Email: sophie.deneve@ens.fr
Course Website: https://lnc2.dec.ens.fr/en/teams/mathematics-neural-circuits/co6-course

Exercise Sheet 4 — 9 April 2019
Please submit your solution next class (16 April 2016)

(1) Covariance and Correlation:
Assume that we have recorded two neurons in the two-alternative-forced choice task discussed in class. We denote the firing rate of neuron 1 in trial \(i\) as \(r_{1,i}\) and the firing rate of neuron 2 as \(r_{2,i}\). We furthermore denote the averaged firing rate of neuron 1 as \(\bar{r}_1\) and of neuron 2 as \(\bar{r}_2\). Let us “center” the data by defining two data vectors

\[
x = (r_{1,1} - \bar{r}_1, r_{1,2} - \bar{r}_1, r_{1,3} - \bar{r}_1, \ldots, r_{1,N} - \bar{r}_1)
\]

\[
y = (r_{2,1} - \bar{r}_2, r_{2,2} - \bar{r}_2, r_{2,3} - \bar{r}_2, \ldots, r_{2,N} - \bar{r}_2)
\]

(a) Show that the variance of the firing rates of the first neuron is \(\text{Var}(r_1) = \frac{1}{N}||x||^2\).

(b) Compute the cosine of the angle between \(x\) and \(y\). What do you get?

(c) What are the maximum and minimum values that the correlation coefficient between \(r_1\) and \(r_2\) can take? Why?

(d) What do you think the term “centered” refers to?

(2) Bayes theorem:
The theorem of Bayes summarizes all the knowledge we have about about the stimulus by observing the responses of a set of neurons, independent of a specific decoding rule.
To get a better intuition about this theorem, we will look at the motion discrimination task again, and compute the probability that the stimulus moved to the left (+) or right (-). For a stimulus \(s = \{+,-\}\), and a firing rate response \(r\) of the neuron, the theorem of Bayes reads

\[p(s|r) = \frac{p(r|s)p(s)}{p(r)}.
\]

Here, \(p(r|s)\) is the probability that the firing rate is \(r\) if the stimulus was \(s\). The respective distribution can be measured and we assume that it follows a Gaussian probability density with mean \(\mu_s\) and standard deviation \(\sigma\),

\[p(r|s) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(r - \mu_s)^2}{2\sigma^2}\right).
\]

The relative frequency with which the stimuli (leftward or rightward motion, + or −) appear is denoted by \(p(s)\), often called the prior probability or, for short, the prior. The distribution \(p(r)\) denotes the probability of observing a response \(r\), independent of any knowledge about the stimulus.

(a) How can you calculate \(p(r)\)?

(b) The distribution \(p(s|r)\) is often called the posterior probability or, for short, the posterior. Calculate the posterior for the stimulus \(s = +\). Sketch the posterior \(p(s = +|r)\), or, for short,
\( p(+) \) as a function of \( r \), assuming a prior \( p(+) = p(-) = 1/2 \). Draw the posterior \( p(-|r) \) into the same plot.

(c) What happens if you change the prior? Investigate how the posterior changes if \( p(+) \) becomes much larger than \( p(-) \) and vice versa. Make a sketch similar to (c).

(d) Let us assume that we decide for leftward motion whenever \( r > \frac{1}{2}(\mu_+ + \mu_-) \). Interpret this decision rule in the above plots. How well does this rule do depending on the prior? What do you lose when you move from the full posterior to a simple decision (=decoding) rule?

(3) **Linear discriminant analysis (advanced):** Let us redo the calculations for the case of \( N \) neurons. If we write \( \mathbf{r} \) for the vector of firing rates observed, then Bayes theorem now reads:

\[
p(s|\mathbf{r}) = \frac{p(\mathbf{r}|s)p(s)}{p(\mathbf{r})}.
\]

We assume that the distribution of firing rates again follows a Gaussian so that

\[
p(\mathbf{r}|s) = \frac{1}{(2\pi)^{N/2}\sqrt{\det C}} \exp \left( -\frac{1}{2}(\mathbf{r} - \mathbf{m}_s)^T C^{-1}(\mathbf{r} - \mathbf{m}_s) \right)
\]

where \( \mathbf{m}_s \) denotes the mean of the density for stimulus \( s \), and \( C \) is the covariance matrix.

(a) Compute the log-likelihood ratio

\[
l(\mathbf{r}) = \log \frac{p(+)\mathbf{r})}{p(-|\mathbf{r})}.
\]

(b) Assume that \( l(\mathbf{r}) = 0 \) is the decision boundary, so that any firing rate vector \( \mathbf{r} \) giving a log-likelihood ratio larger than zero is classified as coming from the stimulus \(+\). Compute a formula for the decision boundary. What shape does this boundary have?

(c) Assume that \( p(+) = p(-) = 1/2 \). Assume we are looking only at two neurons that are uncorrelated so that the covariance matrix is

\[
C = \begin{pmatrix}
\sigma^2_1 & 0 \\
0 & \sigma^2_2
\end{pmatrix}
\]

Sketch the decision boundary for this case.